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STUDY IN MULTIVARIATE APPROXIMATIONS FOR STEERING AND OTHER CONTROL FUNCTIONS IN SPACE FLIGHT OPERATIONS OF SATURN TYPE VEHICLES

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GEORGE C. MARSHALL SPACE FLIGHT CENTER

Prepared by R. J. Vance

R. J. Vance, Senior Mathematician, Principal Investigator

Approved by T L Campbell

T. L. Campbell, Manager Applied Mathematics

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ABSTRACT

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An implementation of the path adaptive guidance mode involves the generation of a manifold of precalculated optimal trajectories through a numerical solution of the Euler-LaGrange equations, followed by approximations of the control variables as functions of the state variables. These approximations, typically, are restricted to polynomials and/or ratios of polynomials. The Gram-Schmidt orthonormalization process has been described in previous reports as an efficient procedure for the generation of these approximations. The studies described in this final report pertain to several problems arising in the general approximation process and to a treatment of the guidance and control problem as a stochastic process. The latter study introduces several concepts of stability for the Euler-LaGrange system of differential equations and suggests problems which will require further study.

Author

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I. INTRODUCTION

The implementation of the concept of Path Adaptive Guidance developed in past reports (References 1, 2 and 3) requires that the control functions be functionally represented and available onboard the vehicle. The first approach to the problem of obtaining representations of these control functions involves the use of a large number of optimized, tabulated trajectories. This manifold of trajectories encompasses all possible disturbances which can affect the flight of the vehicle but which still permit the completion of the mission. Data obtained from these trajectories is then used for approximation of the control functions – pitch and yaw steering angles and cutoff time – for each particular mission. The control functions are approximated by some linear or non-linear combinations of functions of independent state variables. As this approach is developed, several important subproblems can be identified:

1. Necessary and sufficient representation of the control functions as a set or subset of tabulated trajectories.

2. The representation of the control functions by a combination of functions of the state variables – a model of the control functions.

3. Evaluation of the undertermined coefficients of the model by numerical means according to some error criterion using the set of tabulated data as a suitable subset thereof.

4. Using the control functions in simulated flights (or by the application of stochastic processes) to obtain probability statements about the success with which the mission requirements are met.

All of these problems have been examined in some detail. The ensuing discussion will detail some results which have been obtained and the state of current investigations being pursued to resolve other problems.

II. MANIFOLD OF TRAJECTORIES

Typically, tabulated data representing the manifold of trajectories has been generated by numerical solution of the Euler-LaGrange equations for the minimum fuel problem under a range of initial conditions covering that range expected and with which the mission is fulfilled. When the number of stages involved is three or greater, the number of trajectories and, consequently, the amount of tabulated data becomes large. This is due to the rapid increase in number of combinations of disturbances possible. For three dimensional solutions, this situation is aggravated to an even greater extent. Finally, when disturbances are interjected, randomly or regularly, along a trajectory path (rather than only at the staging points), the amount of tabulated data becomes excessive or even prohibitive.

An obvious test for the necessity for inclusion of a particular trajectory in the manifold to be generated is the degree of change in the coefficients of the control functions derived from the data. But this leads to an apparently paradoxical requirement by requiring the obtaining of a judgment as to the effect of the inclusion of a trajectory before the set of trajectories have been computed. This test also implicitly assumes that the model has the true form of the particular control function being sought. If limitations of the vehicle guidance computer constrain the choice of form of control function (e.g., to polynomials or ratios of polynomials), methods used in the design of experiments (Reference 10) provide a possible means to limit the number of trajectories to be computed. The levels of the state variables used in the design of experiments method would be constrained by bounds imposed through applications of astrodynamics in mission planning. The determination of launch "windows" is a well known example of imposing bounds on the levels of state variables.

If the values taken on by the state variables are regarded as having probability distributions, then the above point of view would seem to have some validity. However, an alternate interpretation views each computed trajectory as success or failure with respect to meeting mission requirements. Assuming that the manifold of trajectories has some n-dimensional geometric shape (e.g., a hyper-cone or hyper-rectangle), then trajectories are generated which pass through specified n-dimensional points. Using this manifold of points, the region of accuracy of the control function approximations can be described. This region is given by the ranges of values of the state variables.

A combination of these methods is required for efficiency and completeness. Experimental design yields information as to the spacing of the required points while n-dimensional geometry provides for a description of the shape of the region covered by tabulated data. Methods of numerical approximation generally require both kinds of information.

III. MATHEMATICAL MODELS OF THE CONTROL FUNCTIONS

For several reasons, the mathematical models of the optimal control functions may be constrained to some subclass of functions. As an example of such a constraint, the onboard computer instruction time and/or repertoire may restrict the class of rational algebraic functions to polynomials due to the absence or slow speed of a division instruction. In the past, mathematical models of the optimal control functions have been restricted to polynomials in the position and velocity components (x, y, z, dx/dt, dy/dt, dz/dt) and other measurable state variables; thrust (F), mass flow (dm/dt), and time (t). The assumption of such a model entails problems in both the evaluation of the coefficients of the polynomial terms and the representation of the proper manifold of trajectories for their numerical evaluation. If one assumes a geometrical region R for this manifold – through insight, n-dimensional visualization, etc. – the second problem disappears and the first is simplified.

If a geometrical region R is assumed, then orthogonal polynomials can be generated over this region by the Gram-Schmidt process. This process is described in detail in Progress Reports 3 and 4 where the inner product (f, g) of the functions $f(\underline{x})$, $g(\underline{x})$ over the regions is defined as follows:

$$(f,g) = \int \cdots \int_R fg\,d\underline{x} \tag{1}$$

$$\underline{x} = x_1, x_2, \ldots, x_N \quad , \quad d\underline{x} = dx_1, dx_2, \ldots, dx_N \tag{2}$$

Since geometrical properties of R are assumed, any number of quadrature formulas may be developed to obtain values of inner products. This problem is further discussed in the next section. The Gram-Schmidt orthonormalization also allows the elimination of insignificant terms (References 2 and 3).

R. E. Wheeler in Progress Report No. 4 (Reference 4) has made a direct approach to the problem of the model. While his results hold only for a restricted case, they indicate that a non-linear form is more appropriate than the assumed polynomial model. Evaluation of the coefficients in his model (the ratio of two polynomials) is more difficult than in the case of the polynomial not only from the standpoint of the numerical procedures involved but also from the fact that several sets of coefficients may give satisfactory approximations. That is, the sets of coefficients $\{a_i\}$ and $\{a'_i\}$ may not be unique in the expression (the least "p"th approximation model).

$$\min_{a_i, a'_i} \left\{ \int \cdots \int_R \left[\mathrm{Tan}\,\chi - \frac{a'_0 + a'_1 x_1 + \ldots + a'_m x_m^2}{a_0 + a_1 1 + \ldots + a_L x_L^2} \right]^p d\underline{x} \right\}^{\frac{1}{p}} \tag{3}$$

where χ is one of the control functions being approximated, the x_i are state variables and usually $p = 1, 2, \infty$

If the terms appearing in the rational approximation are not arbitrary but have been derived through considering the equations of motion and the calculus of variations, then the region R should no longer be of critical importance in the determination of $\{a_i\}$ and $\{a_i'\}$. However, R must be large enough to include all relevant disturbances in the state variables. Otherwise, one may obtain sets of coefficients $\{a_i\}$ and $\{a_i'\}$ that yield an approximation with insufficient accuracy over wide ranges of values in the state variables. More precisely, if

$a_i (i = 1, 2, \ldots, m)$

results from using R as the region of integration in Equation (1), then $\left| a_j' - a_j'^* \right| < \epsilon_j^*$ and $\left| a_j - a_j^* \right| \leq \epsilon_j$ for $j = 1, 2, \ldots, m$ for any larger region $R^* \supset R$ of integration. The ϵ_j's and ϵ_j^*'s are small positive constants of the same order as the errors of the computational procedure.

IV. EVALUATION OF THE COEFFICIENTS OF THE SELECTED MODEL

A. LINEAR MODELS AND THE SELECTION OF POINTS

In the case where the selected model is a multivariate polynomial approximation, the Gram-Schmidt Orthonormalization procedure (G.S.O. Procedure) may be applied directly as described on the preceding page and in more detail in References 2 and 3. One of the more important problems which arise in the application of G.S.O. is the selection of points from the manifold of tabulated trajectories. As discussed briefly on the preceding page, the points must be selected in such a way that adequate quadrature formulas can be constructed for the computation of the inner product which at the same time provides a sufficient representation of the integration region. It is at this point that it may be discovered that additional trajectories are required to provide additional points in the n-dimensional integration region. The ensuing discussion describes in detail the generation of particular quadrature formulas for various assumed regions.

One of the basic concepts introduced in the description of the G.S.O. Process (Reference 2) was that of the inner product, (g, f), of two functions g and f. It was defined there as a real valued functional having the properties:

$$\begin{aligned} &\text{i. } (f_1 + f_2, g) = (f_1, g) + (f_2, g) \\ &\text{ii. } (f, g) = (g, f) \\ &\text{iii. } (af, g) = a(f, g) \\ &\text{iv. } (f, f) \geq 0 \end{aligned} \tag{1}$$

For the purpose of approximating functions of many variables (i.e. in the present case, these are the control functions) the inner product of two functions f_1 and f_2 was defined as

$$(f_1, f_2) = \sum_{j=1}^{n} \gamma_j f_{1_j} f_{2_j} \tag{2}$$

where j indicates the value of the function at the jth data point. Immediately the question of how the n data points and weights, γ_j, are selected arises. In the case of control functions, a manifold of trajectories, optimal for some particular mission, define a hyper-volume V over which the inner product must be defined by

$$(f, g) = \sum_{j=1}^{n} \gamma_j f_j g_j \doteq \int \underset{V}{\cdots} \int f\, g dV \tag{3}$$

The data points and weights are selected then to approximate a multiple integral over the volume V. For certain classes of functions, the quadrature formula in (3) will be exact. This condition may be used to determine the points and weights.

For most missions the geometrical shape of the hyper-volume V is unknown. Hence, in order to carry out the integration in (3), some approximation V′ to V in terms of known polytopes must be made from the set of tabulated trajectories by numerical means. The proper selection of the approximation V′ must be made with care since even the choice of relatively simple geometrical shapes can lead to extreme analytic difficulties in the evaluation of the multiple integral,

$$\int \underset{V'}{\cdots} \int fgdV' \tag{4}$$

An approach which has proved helpful in the proper choice of volumes V′ involves a partitioning the volume V into small segments V_i which cover the original volume V. Then one may devise quadrature formulas over approximations V_i' to these smaller and simpler volumes, V_i. To this end we now consider the form of the data obtained in the generation of optimal trajectories.

In computing optimal trajectories for a given mission, the result takes the form of a tabulation.

$$S_i = (\theta_i, \phi_i, T_i, x_{1i}, x_{2i}, \ldots, x_{mi}, t_i)$$

where the subscript i refers to the ith point on a particular trajectory, θ and ϕ are steering angles, while T is time remaining until cutoff and t_i is time from lift-off. θ, ϕ and T are functions of the state variables $x_1, x_2, \ldots, x_m, t$. Approximations of θ, ϕ, and T are required as a basis of an optimal guidance method. A linear combination of functions f_k of the state variables is a type of approximation easily used by existing guidance computers. If the functions f_k are monomials in the state variables the problem of evaluating the approximation in flight is further simplified.

Corresponding to M initial conditions, there will be M trajectories tabulated for a mission. For reasons that will be apparent later, the values of t_i on the M trajectories are the same, i.e., all M trajectories have the (m + 3)tuple S_i given at regular equal intervals τ of time.

By letting $t_0 = 0$, $t_1 = \tau$, $t_2 = 2\tau$, ... at each t_i it is possible to find the maximum and minimum value of each state variable x_1, x_2, ..., x_m by inspecting the components of the M vectors S_i. This is easily accomplished with a subroutine on a digital computer. These values will be used to define the quantities.

$$h_{ji} = \left| \underset{M}{\mathrm{Max}}\, x_{ji} - \underset{M}{\mathrm{Min}}\, x_{ji} \right|_{t = t_i} \tag{5}$$

The symbol $\underset{M}{\mathrm{Max}}$ means the maximum over the M trajectories. The value of h_{ji} is used to define the lengths of an edge of a hyper-rectangle (orthotope).

In order to evaluate the integral in Equation (3), the volume V is divided into segments lying between time t_i and t_{i+2}. Each of these segments is then approximated by a hyper-rectangle R_i with the edge lengths $h_{j,i}$, $j = 1, 2, \ldots, m+1$ parallel to the jth axis. The length $h_{m+1,i}$ is equal to two time intervals

$$h_{m+1,i} = \left| t_{i+2} - t_i \right| = 2\tau$$

Without referring to a particular time t_i, we can discuss quadrature over any hyper-rectangle R with center at

$$\underline{r} = (r_1, r_2, \ldots, r_{m+1})$$

and edges of length h_j. This formula has the form (with error E)

$$\int_R f(\underline{x})\, d\underline{x} = u \sum_{k=1}^{m+1} \left[f\left(\underline{x}^{2k-1}\right) - f\left(\underline{x}^{2k}\right) \right] + Wf\left(\underline{x}^{2m+3}\right) + E \tag{6}$$

where

$$\underline{x} = (x_1, x_2, \ldots, x_m, t) \tag{7}$$

and the superscript indicates the points for the quadrature formula. Thus, x_j^{2k} is the jth component of the vector with superscript 2k. From Reference 5, the components of the 2m+3 vectors used in (6) are given by:

$$x_j^{2k-1} = r_j + \frac{h_j}{2}\left[\frac{\sqrt{3}}{6}\left(1 - \delta_{jk}\right) - \delta_{jk}\right] \tag{8}$$

$$x_j^{2k} = r_j + \frac{h_j}{2}\left[\frac{\sqrt{3}}{6}\left(1 - \delta_{jk}\right) + \delta_{jk}\right] \tag{9}$$

$$x_j^{2m+3} = r_j + \frac{\sqrt{3}}{6} h_j \tag{10}$$

$$W = \prod_{j=1}^{m+1} h_j \tag{11}$$

$$u = \frac{\sqrt{3}}{6} \prod_{j=1}^{m+1} h_j \tag{12}$$

where δ_{jk} is the Kronecker delta. This quadrature formula will be exact for second degree polynomials.

A much simpler,, but more erroneous, quadrature formula results from the direct application of the mean value theorem. In this case the integral over a hyper-rectangle R would take the form

$$\int \cdots \int_R f(\underline{x})\, d\underline{x} = Wf(\underline{r}) + E_2 \tag{13}$$

The errors E_1 and E_2 are composite ones resulting from approximating the volume V by V′, $f(\underline{x})$ not being a function for which the quadrature is exact, and the computational roundoff errors.

B. NON-LINEAR APPROXIMATIONS – RATIONAL FORMS

If the mathematical model of the control functions assumes a rational form, then several methods are immediately available to determine the coefficients $\{a_j\}$, $\{a_j'\}$. Moreover, these methods have been programmed and used. They are:

- Direct use of orthonormalization by removing the denominator of the model through multiplication.
- A modified Newton-Raphson (Gauss-Seidel) method combined with steepest descent with a least squares error criterion.
- A modified Linear Programming system.

The orthonormalizational approach to rational approximation was described in Reference 2, p. 233. In that approach, the model assumed was:

$$\chi \approx \frac{P_1(\underline{x})}{1 + P_2(\underline{x})} \tag{14}$$

where P_1 and P_2 were polynomials of a given degree in the state variables $x_1, x_2, \ldots, x_n$. Then the linear form

$$\chi \approx P_1(\underline{x}) - \chi P_2(\underline{x}) \tag{15}$$

was used to determine the coefficients of P_1 and P_2 by an iterative method. This method is useful since no initial guess must be made for values of the coefficients. It may happen that even after iteration, the coefficients are not too near the value which minimizes

$$\sum_{j=1}^{M} \left[\chi_j - \frac{P_1(\underline{x}_j)}{1 + P_2(\underline{x}_j)} \right]^2 = Q(\chi, \underline{x}) \tag{16}$$

where the subscript indicates the value of variables at the j-th data point. If this proves to be the case, the modified Newton-Ralphson method may be used with the coefficients obtained from Equation (16) as the initial guess.

The modified Newton-Raphson method has been described in great detail in Reference 6. Essentially, it consists of linearizing the non-linear approximation by a Taylor series expansion about the coefficients. Letting f be the approximation of the control function χ, we have

$$\chi - f(\underline{x}, a_0, \ldots, a_m, a_0', \ldots, a_L') \approx$$

$$\approx \frac{\partial f}{\partial a_0} \Delta a_0 + \ldots + \frac{\partial f}{\partial a_m} \Delta a_m + \frac{\partial f}{a_0'} \Delta a_0' + \ldots + \frac{\partial f}{\partial a_L'} \Delta a_L' \tag{17}$$

where the subscripts on the a_k and a_k' refer to those used in Equation (1). In other words, the error e is expressed in terms of differences Δa_k, $\Delta a_k'$ in the coefficients a_k and a_k'. Starting out with an initial guess of the coefficients a_k, a_k', we may use the usual least squares method of an orthonormalization code to obtain estimates of the Δa_k, $\Delta a_k'$ from the equations:

$$e_j = \left[\chi_j - f(\underline{x}_j, a_0, \ldots, a_L') \right]$$

$$e_j \approx \sum_{i=0}^{m+L+1} \frac{\partial f_j}{a_i} \Delta a_i \qquad j = 1, 2, \ldots, n$$

$$\left(a_{m+1+i} = a_i' \right)$$

where the subscript j refers to the j-th data point. At each of p steps, a correction is made to the coefficients a_k, a_k'

$$a_{k,p+1} = a_{k,p} + h\Delta a_{k,p+1}, \quad 0 < h \leq 1 \tag{19}$$

Equation (19) results after (p+1) steps in estimating the coefficient a_k. The value of h is determined by some modified method of steepest descent. One advantage of a computer program using this procedure is that the approximation need not be restricted to a rational form in the state variables. The method based on linear programming (Progress Report No. 4, p. 287) would seem to have such a restriction.

Whenever the approximation is a rational form, there is the possibility that several of the zeros of the denominator may occur within the range of values taken on by the state variables during a flight. The determination of the zeros of a polynomial in many variables is a problem of extraordinary difficulty. However, conditions may be placed on the coefficients of a polynomial that are sufficient to guarantee its being positive for all or some real values of the variables. Trivial examples would be requiring the polynomial be a positive definite quadratic form or that all the coefficients be positive with the variables taking on positive values. More general results appear in a paper by Perlin (Reference 7). There would seem to be only minor difficulties in altering the denominator of a rational approximation to insure that it did not become zero in the domain of the control function.

V. A STOCHASTIC VIEW OF PATH ADAPTIVE GUIDANCE

Once the approximation of the optimal control functions are sufficiently accurate in terms of the error criterion used, it is necessary to assess the worth of the approximations by simulated flights. Ideally, we would like to dispense with the simulation and make a probability statement concerning the control system's properties from the magnitude of the errors of the approximation at the data points used. These properties would include such items as accuracy in position, velocity, time, fuel used in excess of a true optimal flight path, and sensitivity to errors in input from the vehicle's sensors.

We leave the deterministic point of view for the purpose of making probability statements about the success of a set of approximations of the control functions. The successive states of a space vehicle along its flight path can be regarded as a Markov process or more exactly a discrete, finite, multiple Markov vector chain. A finite Markov chain is defined as a Markov process whose random variables $\underline{y}_t$ (vector valued) can assume values in a certain set of vectors $\{\underline{Y}_t\}$ $t = 1, 2, \ldots, N$ with probability 1.

Definition: (Reference 8) Let T be an index set (also called parameter set), then a (simple) Markov process is a process $\{y_t, t \epsilon T\}$ such that for any integer $n \geq 1$, if $t_1 < \ldots < t_n$ are parameter values, the conditional probabilities of x_{t_n} given x_{t_1}, $x_{t_2}, \ldots, x_{t_{n-1}}$ are the same as the conditional probability given $x_{t_{n-1}}$ in the sense that for each real number V

$$\Pr\left\{x_{t_n} \leq V \middle| y_{t_1}, \ldots, y_{t_{n-1}}\right\} = \Pr\left\{y_{t_n} \middle| y_{t_{n-1}}\right\}. \tag{1}$$

A Markov process is a multiple process if

$$\Pr\left\{y_{t_n} \leq V \middle| y_{t_1}, \ldots, y_{t_{n-1}}\right\} = \Pr\left\{y_{t_n} \leq V \middle| y_{t_{n-u}}, \ldots, y_{t_{n-1}}\right\}. \tag{2}$$

A random process is a family of random variables $\{x_t\}$.

Less exactly, a simple Markov process is a process in which the value that y_{t_n} takes on depends only on the value that $y_{t_{n-1}}$ assumed. In the multiple Markov process, y_{t_n} depends on the values of the u previous random variables $y_{t_{n-u}}, \ldots, y_{t_{n-1}}$. One might say that Markov processes, when referred to particle dynamics, are generalizations of Newtonian mechanics. The multiple Markov process is more of a generalization than the simple Markov process. The index set T or parameter set is time in most applications.

In the present case

$$T = \left\{ t_j \middle| t_L < t_j < t_c \right\} \tag{3}$$

where t_L is the time of earliest possible liftoff and t_c is latest possible cutoff for a successful completion of the mission. One also could use the time interval over which control is possible for all possible variations of the state of the vehicle during its flight.

After suitable simulation, an estimation of the probability of successful completion of the mission, with the given control functions, can be made. This may take the form, in the simplest approach, of recording the flight as a success or failure by asking: Was the state of the vehicle one of those classified as successful in the time interval $\left[t_{s_1}, t_{s_2}\right]$? For computer use, the idea of an absorbing barrier can be used. If a criterion can be formulated explicitly, in the form of a computer program subroutine, to distinguish between the two classes of flights then there are two absorbing barriers which terminate the simulated flights: one defined for a successful mission and the other for an unsuccessful completion.

A more suitable approach, in terms of ease of execution, would be to fix some nominal point with a region defined about it as one into which a trajectory must enter to be a success. This region can usually be given in the form of an ellipsoid

$$Q_N = \sum_{i=1}^{n} b_i(x_i - x_{Ni})^2 \leqq k \tag{4}$$

where the x_{Ni} are fixed values of the state variables (the coordinates of the point) and the x_i are components of a vehicle's state vector $\underline{x}$ previously referred to. Assuming a normal distribution for each x_i, we would like to make a probability statement

$$\Pr\left[\sum_{i=1}^{n} b_i(x_i - x_{Ni})^2 \leq k\right] = p \tag{5}$$

This states the probability of a successful mission given the set of approximations of the optimal control functions. The most direct way of obtaining values of p in equation (9) is by the numerical integration of distribution function of

$$\sum_{i=1}^{n} b_i(x_i - x_{Ni})^2$$

or of its inverse Fourier transform. The problem of numerically integrating the inverse Fourier transform of the distribution of Q_n has been solved by Imhof (Reference 9).

Q_n is transformed into a linear combination of independent noncentral chi-square variables

$$Q_n \sum_{r=1}^{n} \lambda_r x_{h_r}^2 ; \delta_r^2 \tag{6}$$

where

$$x_h^2, \delta^2 = (x_1 + \delta)^2 + \sum_{i=2}^{h} x_i^2$$

The x_i are independently distributed N(0, 1), i. e., unit, central normal variates. Making use of the fact that

$$\Pr [Q_n \leq k] = 1 - \Pr [Q_n > k] \qquad [7]$$

we arrive, through involved procedures, at the equation

$$\Pr [Q_n > k] = \frac{1}{2} + \frac{1}{\pi} \int_0^{\infty} \frac{\sin \theta (u)}{u \, \Omega (u)} du$$

where

$$\theta (u) = \frac{1}{2} \sum_{r=1}^{n} \left[h_r \tan^{-1} (\lambda_r u) + \delta_r^2 \lambda_r u \;\; (1 + \lambda_r^2 u^2) \right] - \frac{1}{2} ku$$

$$\Omega (u) = \prod_{r=1}^{n} (1 + \lambda_r^2 u^2)^{\frac{h_r}{4}} \exp \left\{ \frac{1}{2} \sum_{i=1}^{N} (\delta_r \lambda_r u)^2 \;\; (1 + \lambda_r^2 u^2) \right\}$$

The parameters λ_r, δ_r may be estimated from the data obtained in simulated flights, while h_r is known exactly and is usually called the "degrees of freedom" of the chi-square distribution. The approach followed here need not be restricted to an ellipsoid about some point $\underline{x}_N$. The "region of success" may be extended to ellipsoids about several points.

VI. RECOMMENDATIONS FOR FUTURE WORK*

A problem of fundamental importance in the development of space flight is that of control and stability of the trajectory of a space vehicle. Because of the large number of factors which influence the trajectory, it is difficult, if not impossible, to construct a vehicle whose trajectory will be, in any reasonable sense, inherently stable. It is therefore, necessary to introduce control devices in order to make corrections in the trajectory which will enable some acceptable criterion of stability to be satisfied. In recent years a considerable amount of research in the area of control and stability has been done by engineers and mathematicians. In the main, these studies have been carried out within the framework of the theory of differential equations. A survey of the relevant literature was conducted in order to study and characterize the various approaches to control and stability that have been formulated; with particular attention being given to the large number of Soviet contributions to the subject. The approaches can be characterized as deterministic, stochastic, and mixed, i. e., deterministic and stochastic. While considerable progress has been made toward the solution of certain special cases of control and stability, it is clear that much work remains to be done before a satisfactory theory is developed which will be applicable to a realistic treatment of the control and stability of space vehicle trajectories.

The approach we propose is based on the study of the fundamental Euler-LaGrange equations (E-L eqs) within the framework of the theory of random differential equations. Random solutions of the E-L eqs will arise if we solve the E-L eqs with either (1) random initial conditions, (2) equation parameters subject to random variations, or (3) a combination of (1) and (2). In each of the above cases a family of solutions (realizations, or trajectories) of the E-L eqs will be generated, the family of solutions generated depending, of course, on the nature of the probability distribution imposed. While it is relatively easy to write down random analogues of the E-L eqs, the analytical difficulties involved in obtaining the random solution and its probability distribution are considerable. In view of these difficulties, we propose the use of computer methods in the study of the random solutions generated by the E-L eqs, in each of the three cases mentioned above, when various probability distributions are assumed. Computer methods will not only enable us to generate a large number of solutions, but will permit the computation of moments and other statistics associated with the random solutions.

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As concrete problems, we propose that the following cases be investigated:

1. One-stage, fixed target, with random initial conditions.

2. One-stage, fixed target, with random variation of equation parameters.

3. Combination of 1 and 2.

4. Two-stage, fixed target, with random initial conditions and random variation of parameters. In this case the positions at the end of the first stage will form a set of random initial conditions for the solution of the equations in the second stage.

5. Three-stage, fixed target, with conditions the same as 4.

For all of the above cases we suggest that as a first approximation a Gaussian distribution of the initial conditions and equation parameters be assumed.

Several criteria of trajectory stability have been formulated. The first is what might be classed as <u>stage stability</u>, as this involves the probability that the trajectory lie within a given subspace of the state space throughout a given stage. The second is what might be termed <u>stability with respect to target</u>, as this involves the probability that at the end of the flight the trajectory reach a given region of the state space (see Section V, equation 4). In the case where the state vector has three position components and three velocity components, the condition to be imposed is that the trajectory reach a spere of specified radius in 6-dimensional state space, the center of which is the desired target. A third type of stability can be introduced which involves both stage stability and stability with respect to target. Analogous concepts of <u>mean stage stability</u> and <u>mean stability with respect to target</u> can also be defined.

When the numerical solutions are available it is suggested that those cases which yield solutions or trajectories with desired characteristics be formulated as random differential equations and studied analytically. In this way we can obtain more rigorous results concerning the stochastic processes generated by random solutions of the E-L eqs. At this stage control theory can be introduced, for the problem can then be formulated as the control and stability of the realization of a concrete stochastic process.

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